Vector analysis and Maxwell's equations: Questions.

I. VECTOR ANALYSIS

1. What is the geometrical meaning of a dot product between two vectors?

2. What is the geometrical meaning of a cross product between two vectors?

3. According to Rule 1 a cross and a dot can be interchanged:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

   Explain the geometrical interpretation of this identity.

4. According to Rule 2:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

   Explain why the resulting vector must lie on the plane formed by the vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., it is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

5. What is a scalar field? Give some examples.

6. What is a vector field? Give some examples.

7. Explain the physical meaning of divergence of a vector field.

8. Explain the physical meaning of rotation or curl of a vector field.

9. Describe how to use the chain rule when applying a nabla operator on a product of two quantities consisting of scalar and vector fields.

10. Describe how to calculate a line integral and what is the meaning of a line integral?

11. Describe how to calculate a surface integral and what is the meaning of a surface integral?

12. Describe how to calculate a volume integral.

13. Explain how to understand Stokes' formula.

14. Explain how to understand Gauss' formula.

15. What is a conservative field?

16. Show or verify that a vector field that can be obtained as a gradient of a scalar field must be conservative.

17. Explain why the work done along a closed loop is zero if the field is conservative.

18. Explain why the rotation or curl of a conservative field must be zero.

II. MAXWELL'S EQUATIONS

1. What is the electric field arising from a point charge q? Sketch how the electric field lines look like when the charge is positive and when the charge is negative. What is the meaning of these lines?

2. Sketch how the electric field lines look like when there are two charges $+q$ and $-q$ separated in space.

3. By considering a point charge q and the electric field flux across the surface of a sphere of radius R centered at the charge, derive Maxwell's first equation (Coulomb's law).

4. Derive the corresponding differential form of Maxwell's first equation.

5. What is the physical meaning of the second Maxwell's equation, $\nabla \cdot \mathbf{B} = 0$?

6. What is Faraday's law? Illustrate it by an example.

7. Derive Maxwell's third equation.

8. What is Lenz' law? Explain how it works.

9. What is Biot-Savart's law?

10. What is Ampere's law and under what condition it is valid?

11. Illustrate the problem with Ampere's law. What is missing in Ampere's law?

12. What is the physical meaning of the continuity equation?

13. Obtain Maxwell's fourth equation by modifying Ampere's law and imposing the continuity equation.

14. Derive the electromagnetic wave equations in vacuum.

15. What is the general form of solutions to the wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

16. From the electromagnetic wave equations, deduce that the speed of light in vacuum is given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

17. Consider a plane-wave solution to the electromagnetic wave equations. Prove that the electric field, the magnetic field, and the propagation vector must be perpendicular to each other.

18. For the plane wave, sketch how the electric and magnetic fields look like in relation to the direction of propagation.

19. Compare the magnitude of the magnetic field of a typical radio wave with the earth's magnetic field and a small bar magnet.

20. How are Maxwell's equations modifed in a transparent material such as glass?