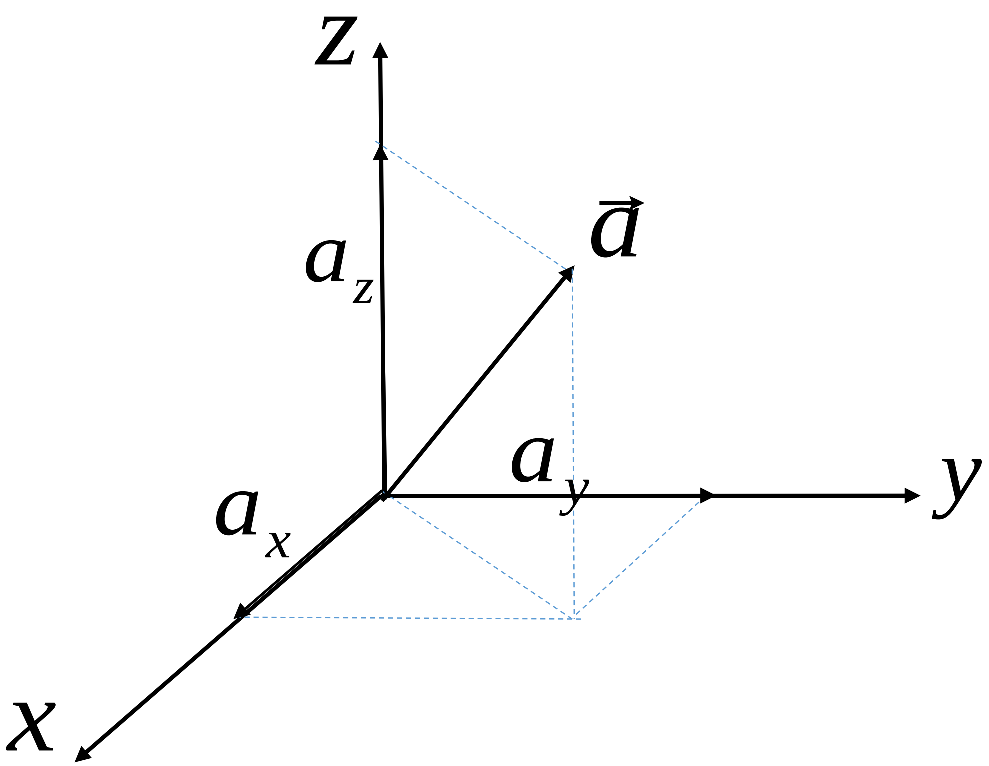
Review of vector analysis

Vectors



In Cartesian coordinate system
a vector can be decomposed into its
components along the *x, y,* and *z* axes.

$$\vec{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z$$

$\hat{e}_x, \hat{e}_y, \hat{e}_z$ are unit vectors along the *x, y,* and *z* axes.

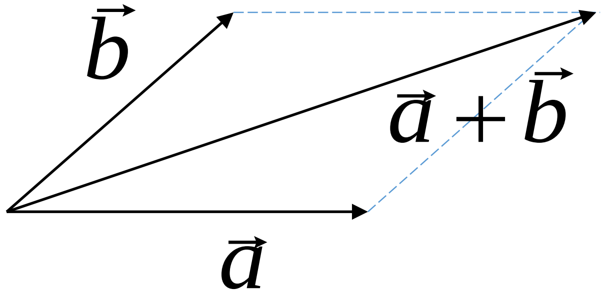
They are *perpendicular* to each other.

Addition of two vectors

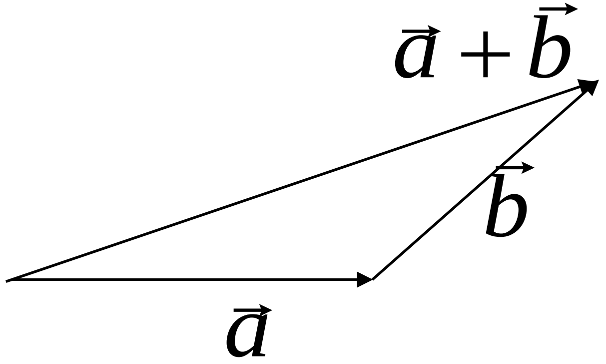
$$\vec{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z$$

$$\vec{b} = b_x \hat{e}_x + b_y \hat{e}_y + b_z \hat{e}_z$$

$$\vec{a} + \vec{b} = (a_x + b_x)\hat{e}_x + (a_y + b_y)\hat{e}_y + (a_z + b_z)\hat{e}_z$$



or

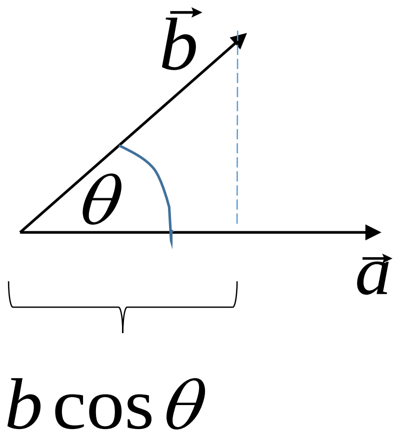


Scalar product (Dot product) between two vectors

$$\vec{a} = a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z$$

$$\vec{b} = b_x\hat{e}_x + b_y\hat{e}_y + b_z\hat{e}_z$$

Geometrical meaning:



$$\vec{a}\cdot\vec{b} = ab\cos\theta$$

$a = |\vec{a}| = \text{magnitude of } \vec{a}$

$\vec{a}\cdot\vec{b} = 0$ if $\vec{a}$ and $\vec{b}$ are perpendicular to each other

$$(\theta = \frac{\pi}{2} \rightarrow \cos\theta = 0)$$

In Cartesian coordinate:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

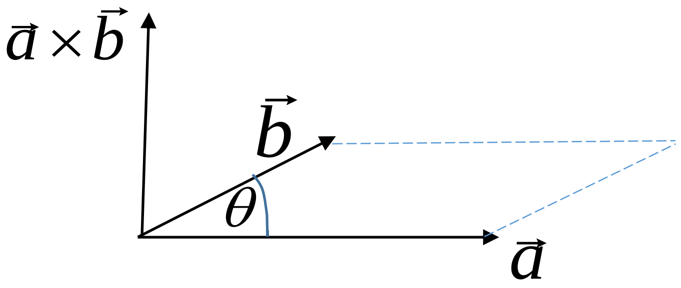
A dot product is a *number* (scalar)

because $\hat{e}_x\cdot\hat{e}_y = \hat{e}_x\cdot\hat{e}_z = \hat{e}_y\cdot\hat{e}_z = 0$

Cross product (Vector product) between two vectors

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\vec{e}_x + (a_z b_x - a_x b_z)\vec{e}_y + (a_x b_y - a_y b_x)\vec{e}_z$$

Geometrical meaning:



A cross product is a *vector* perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$

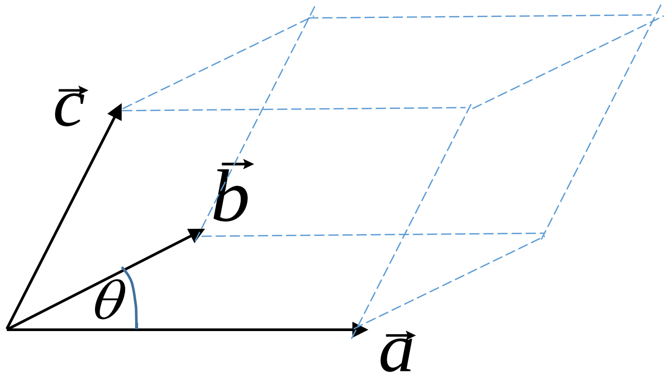
$|\vec{a}\times\vec{b}| = |ab\sin\theta|$ = area of parallelogram

$\vec{a}\times\vec{b} = 0$ if $\vec{a}$ and $\vec{b}$ are parallel to each other

$(\theta = 0 \rightarrow \sin\theta = 0)$

Note that: $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

Rule 1: $(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$ The cross (x) and the dot (.) can be interchanged



$$\left.\begin{array}{l}(\vec{a}\times\vec{b})\cdot\vec{c}\\(\vec{b}\times\vec{c})\cdot\vec{a}\\(\vec{c}\times\vec{a})\cdot\vec{b}\end{array}\right\}=\text{volume of parallelepiped}$$

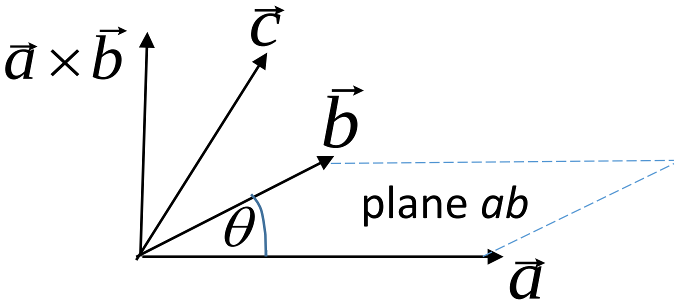
This rule is not difficult to remember.
The key point is to keep the cyclic order: abc, bca, cab, whereas acb, bac, cba, will introduce a minus sign.

Note that $\vec{a}\times\vec{b}\cdot\vec{c}$ has no meaning or ambiguous.

Rule 2: $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$

$(\vec{a}\times\vec{b})\times\vec{c}$ must lie on plane *ab.*

The vector in the middle (***b***) has a positive coefficient



Note that: $\vec{a}\times\vec{b}\times\vec{c}$ has no meaning or ambiguous because

$$\underbrace{(\vec{a}\times\vec{b})\times\vec{c}}_{\text{on plane } ab}\neq\underbrace{\vec{a}\times(\vec{b}\times\vec{c})}_{\text{on plane } bc}$$

Geometric interpretation:

$\vec{a}\times\vec{b}$ must be perpendicular to plane *ab* formed by $\vec{a}$ and $\vec{b}$.

$\vec{c}$ can be decomposed as $\vec{c}=\vec{c}_{\perp}+\vec{c}_{=}$

($\vec{c}_{\perp}$: perpendicular to plane *ab*; $\vec{c}_{=}$: on plane *ab*)

$$(\vec{a}\times\vec{b})\times\vec{c}=\underbrace{(\vec{a}\times\vec{b})\times\vec{c}_{\perp}}_{0}+\underbrace{(\vec{a}\times\vec{b})\times\vec{c}_{=}}_{\text{on plane } ab}$$

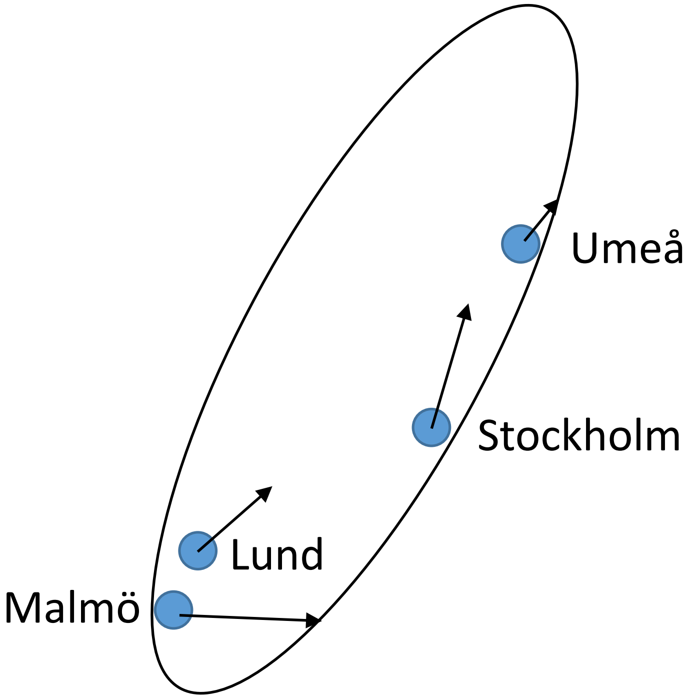
Scalar and vector fields

A scalar field is a function of position in space. It is ***a scalar or a number***.

For example, a temperature field $T(x,y,z)$ tells us the temperature at point (x,y,z) in space.

A vector field is also a function of position in space but it is a vector, i.e., it has a ***magnitude and direction***.

For example, a wind velocity field on a weather chart: $\vec{v}(x,y,z)$



Note that both scalar and vector fields may depend on additional variables such as time:

$$\vec{v}(\vec{r},t)$$

Nabla operator (gradient operator)

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

It has three components and behaves as a vector.

Nabla operator on a scalar field: $\phi(\vec{r})$

$$\nabla \phi(\vec{r}) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \frac{\partial \phi}{\partial x}\vec{e}_x + \frac{\partial \phi}{\partial y}\vec{e}_y + \frac{\partial \phi}{\partial z}\vec{e}_z$$

Gradient of a scalar field is a ***vector***.

It describes the rate of change of the scalar field along the *x, y,* and *z* directions.
It provides information about the rate of change of the scalar field in ***any direction.***

The rate of change in an arbitrary direction $\hat{n}$ is given by

$$\hat{n} \cdot \nabla \phi(\vec{r}) = \frac{\partial \phi}{\partial x}(\hat{n} \cdot \vec{e}_x) + \frac{\partial \phi}{\partial y}(\hat{n} \cdot \vec{e}_y) + \frac{\partial \phi}{\partial z}(\hat{n} \cdot \vec{e}_z)$$

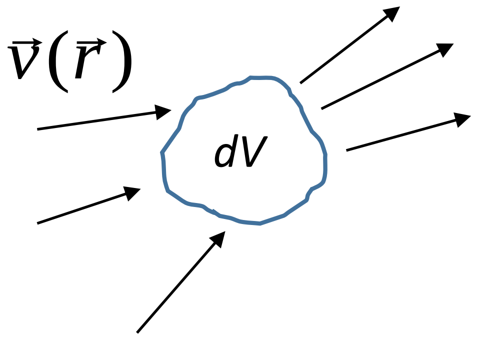
Nabla on a vector field: Divergence

$$\nabla \cdot \vec{v}(\vec{r}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Divergence is a ***scalar (number)***, not a vector.

The "dot" is very important.
$\nabla \vec{v}$ has ***no meaning***.

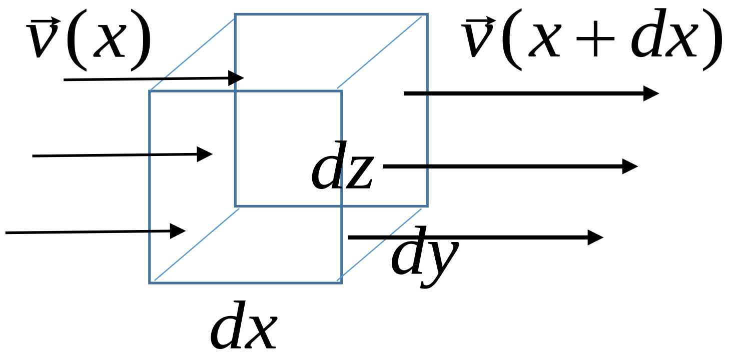
Physical meaning: the ***net flux*** (field lines) going out of a small volume *dV*.



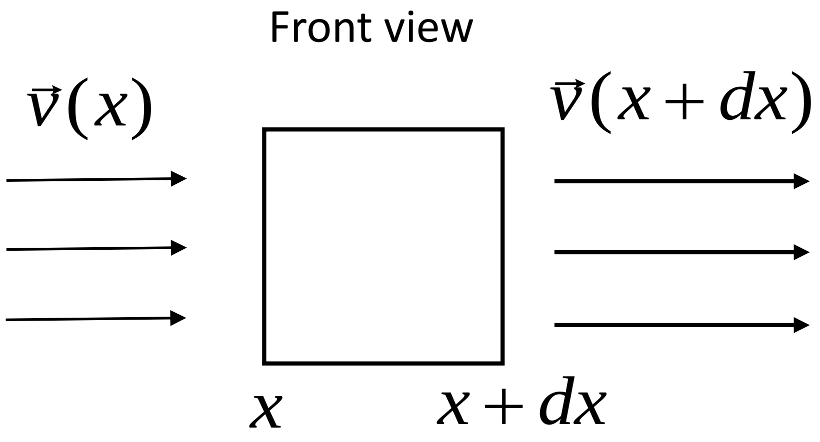
$$\left(\nabla \cdot \vec{v}(\vec{r})\right) dV = \text{net flux}$$

Net flux is zero if there is no source inside the small volume (incoming flux = outgoing flux)

Net flux is finite is there is a source inside the small volume.



(For example:
water flow in a river
through a volume
of size *dV* = *dx dy dz*)



$$\text{net flux} = \left[\vec{v}(x+dx) - \vec{v}(x)\right] dydz$$

surface area passed through
by the filled lines

$$= \underbrace{\frac{v_x(x+dx) - v_x(x)}{dx}}_{\frac{\partial v_x}{\partial x}} \underbrace{dxdydz}_{dV}$$

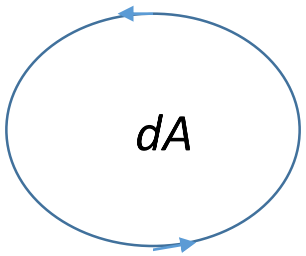
Considering all directions
:

$$\boxed{\text{net flux} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dV = \nabla \cdot \vec{v}(\vec{r})\, dV}$$

Nabla cross a vector field: Curl or Rotation

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \vec{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \vec{e}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \vec{e}_z$$

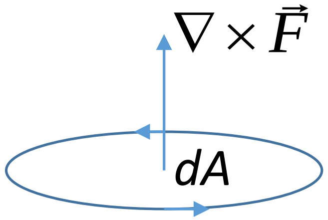
The “work” done by the field around a small loop is equal to the rotation multiplied by the area of the loop.



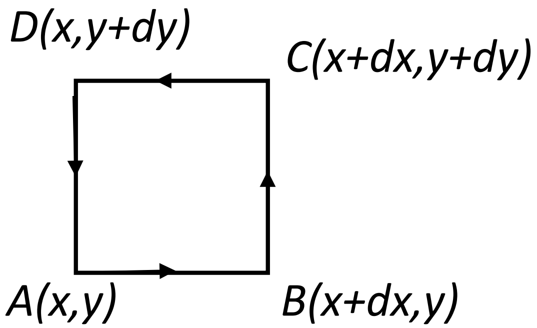
Work done by the field around a loop of area *dA*:

$$(\nabla \times \vec{F}) \cdot d\vec{A}$$

$(\nabla \times \vec{F})$: perpendicular to the loop plane



Consider the work done by a vector field ***F*** around a small loop on the *x*-*y* plane:



work done along *AB* = $F_x(x,y)dx$

work done along *BC* = $F_y(x+dx,y)dy \quad = F_y(x,y)dy + \frac{\partial F_y}{\partial x}dxdy$

work done along *CD* = $-F_x(x+dx,y+dy)dx \quad = -F_x(x,y)dx - \frac{\partial F_x}{\partial y}dxdy$

work done along *DA* = $-F_y(x,y+dy)dy \quad = -F_y(x,y)dy$

(only keep first-order terms in *dx* and *dy)*

work done along the loop= $\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)dxdy$

z-component of $\nabla \times \vec{F}$

For an arbitrary loop of area *dA*, the work done along the loop by the field is given by $(\nabla \times \vec{F}) \cdot d\vec{A}$

Laplacian operator on a scalar field

$$(\nabla\cdot\nabla)\phi(\vec{r}) = \nabla^2\phi(\vec{r}) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

This is a scalar or a number

Nabla operating on several quantities

How to calculate $\nabla\cdot(\phi\,\vec{v}) = ?$

Use chain rule:

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\,\frac{dg}{dx}$$

$$= \left[\left(\frac{d}{dx}\right)_f + \left(\frac{d}{dx}\right)_g\right](fg)$$

split the derivative into two parts:
one acting on *f* only and another on *g* only

Let us apply the chain rule to $\nabla \cdot (\phi\, \vec{v})$

$$\nabla \cdot (\phi\, \vec{v}) = (\nabla_{\phi} + \nabla_{v}) \cdot (\phi\, \vec{v})$$

Split the derivative (nabla)

$$= \nabla_{\phi} \cdot (\phi\, \vec{v}) + \nabla_{v} \cdot (\phi\, \vec{v})$$

$$= (\nabla_{\phi} \phi) \cdot \vec{v} + \phi (\nabla_{v} \cdot \vec{v})$$

Nabla on ϕ must be a vector and nabla on ***v*** must be divergence

$$= (\nabla \phi) \cdot \vec{v} + \phi (\nabla \cdot \vec{v})$$

Drop the subscripts ϕ and *v*

Another example: $\nabla\times(\vec{a}\times\vec{b}) = (\nabla_a + \nabla_b)\times(\vec{a}\times\vec{b})$ Split the nabla

$$= \nabla_a\times(\vec{a}\times\vec{b}) + \nabla_b\times(\vec{a}\times\vec{b})$$

Consider each term: $\nabla_a\times(\vec{a}\times\vec{b}) = \alpha\vec{a} - \beta\vec{b}$

Use rule 2: treat nabla as a vector.
Recall that a triple cross product produces a vector on a plane formed by the vectors in the bracket and the vector in the middle (***a***) has a positive sign.

The coefficients α and β must be scalars:

$$\alpha = \begin{cases} \nabla_a\cdot\vec{b} & \rightarrow \text{makes no sense} \\ \vec{b}\cdot\nabla_a \end{cases}$$

$$\beta = \begin{cases} \nabla_a\cdot\vec{a} \\ \vec{a}\cdot\nabla_a & \rightarrow \text{makes no sense because the nabla should act on } \boldsymbol{a}\text{, not on } \boldsymbol{b} \end{cases}$$

$$\left.\begin{aligned} \nabla_a\times(\vec{a}\times\vec{b}) &= (\vec{b}\cdot\nabla_a)\vec{a} - (\nabla_a\cdot\vec{a})\vec{b} \\ \nabla_b\times(\vec{a}\times\vec{b}) &= (\nabla_b\cdot\vec{b})\vec{a} - (\vec{a}\cdot\nabla_b)\vec{b} \end{aligned}\right\} \quad \nabla\times(\vec{a}\times\vec{b}) = (\vec{b}\cdot\nabla + \nabla\cdot\vec{b})\vec{a} - (\nabla\cdot\vec{a} + \vec{a}\cdot\nabla)\vec{b}$$

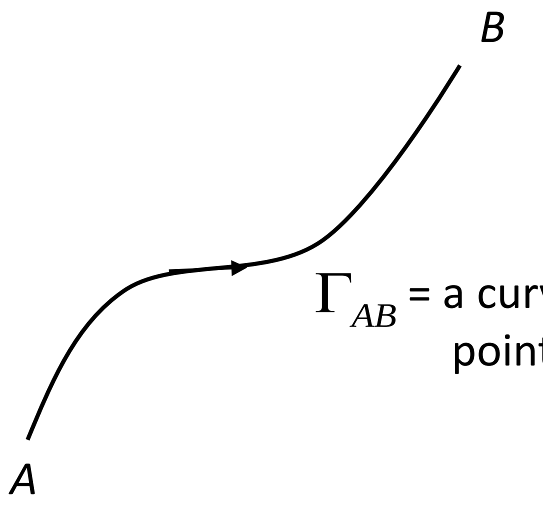
Integrals

1) Line integral
2) Surface integral
3) Volume integral

Integrals are scalars or numbers.

Line integral

$$I_{A\to B} = \int_{\Gamma_{AB}} d\vec{r} \cdot \vec{F}(\vec{r})$$



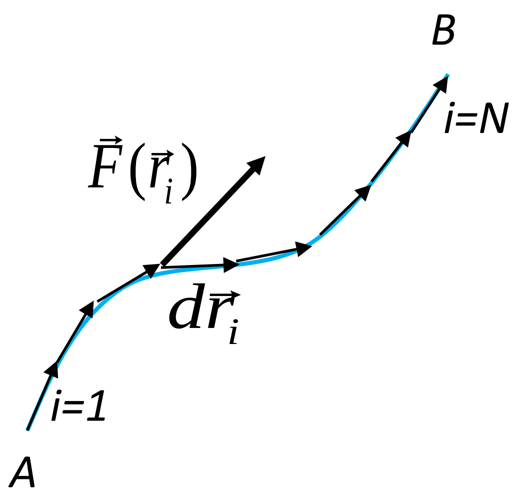
Γ_{AB} = a curve in space from point *A* to point *B*

- The "dot" is very important. Without the dot, the expression makes no sense.
- A line integral must be defined with respect to a given curve and the direction is important:

$$I_{A\to B} = -I_{B\to A}$$

- If ***F(r)*** is a force field, the line integral can be thought of as the work done by the field from point *A* to point *B*.

Meaning of line integral



Divide the curve into *N* small segments
and sum the work done on each segment:

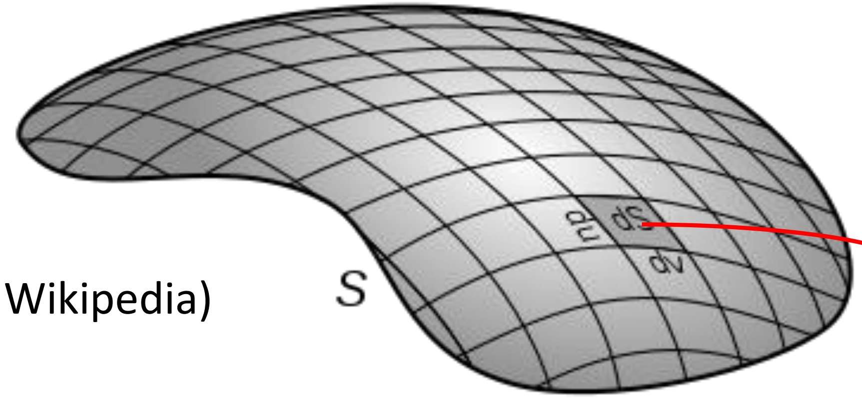
$$\int_{\Gamma_{AB}} d\vec{r} \cdot \vec{F}(\vec{r}) \approx \sum_{i=1}^{N} \underbrace{d\vec{r}_i \cdot \vec{F}(\vec{r}_i)}_{\text{work done in segment } i}$$

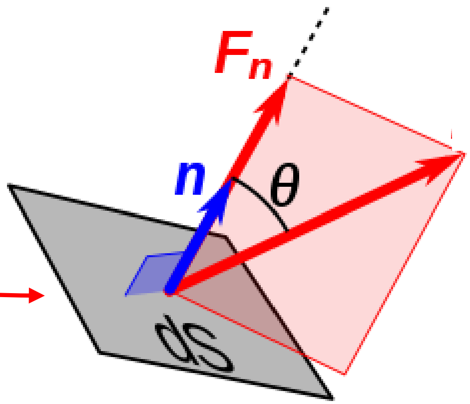
As *N* is increased, the sum approaches the exact integral.

Surface integral

$$\int_S d\vec{S}\cdot\vec{F}(\vec{r}) \approx \sum_{i=1}^{N} d\vec{S}_i\cdot\vec{F}(\vec{r}_i)$$

(Pictures from Wikipedia)





$\vec{F}(\vec{r}_i)$ The field at $\vec{r}_i$ piercing through the surface

Divide the surface into small segments *dS.*

A small segment of the surface *S* and its contribution to the surface integral is given by

$$d\vec{S}_i\cdot\vec{F}(\vec{r}_i) = dS_i F(\vec{r}_i)\cos\theta = dS_i F_n(\vec{r}_i)$$

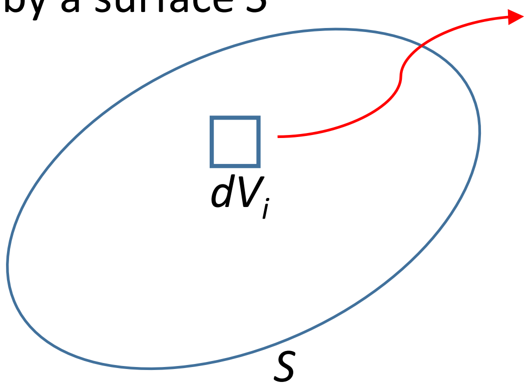
$d\vec{S}_i$ is a ***vector*** at position $\vec{r}_i$ on the surface *S* with magnitude dS_i (area of the small segment) and normal (perpendicular) to the surface.

Volume integral over a scalar field

$$\int_{V(S)} dV \phi(\vec{r}) \approx \sum_{i=1}^{N} dV_i \phi(\vec{r}_i)$$

A volume *V* enclosed by a surface *S*

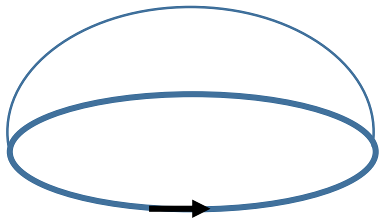
Divide the volume into *N* small segments



A small volume segment of size dV_i centred at $\vec{r}_i$

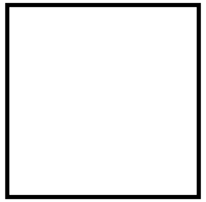
Stokes formula

$$\oint_{C(S)} d\vec{r} \cdot \vec{F}(\vec{r}) = \int_{S} d\vec{S} \cdot (\nabla \times \vec{F}(\vec{r}))$$

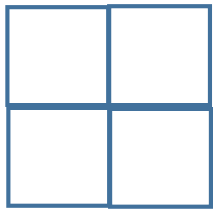


The surface is arbitrary, as long as it encloses the loop *C*

Consider a closed loop

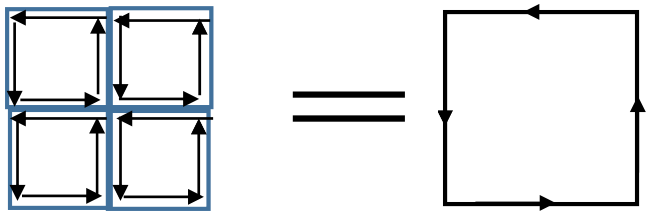


→ divide into small segments



Recall that the work done by the vector field ***F*** around a small loop is equal to the rotation multiplied by the area of the loop:

$$d\vec{r} \cdot \vec{F}(\vec{r}) = d\vec{S} \cdot (\nabla \times \vec{F}(\vec{r}))$$

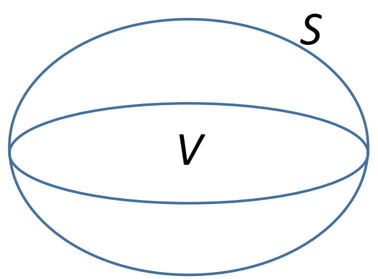


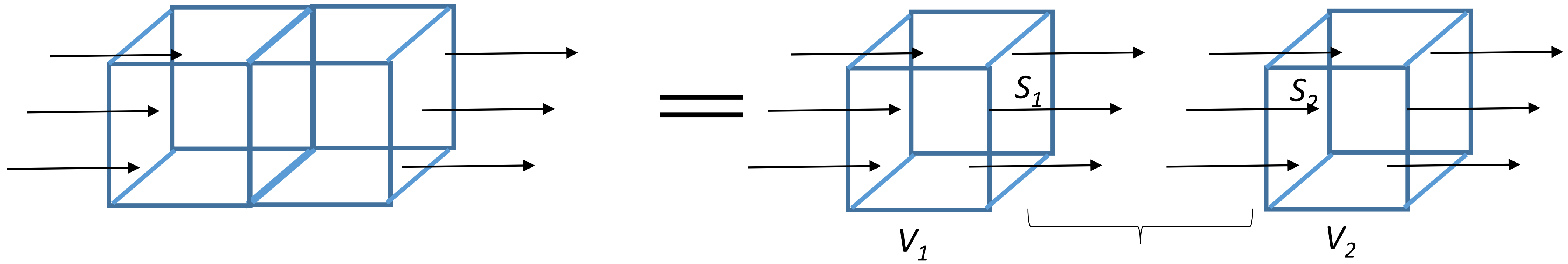
When summing over the small segments, contributions from the inner paths cancel out.

Gauss formula

$$\int_{S(V)} d\vec{S}\cdot\vec{F}(\vec{r}) = \int_{V(S)} dV\ \nabla\cdot\vec{F}(\vec{r})$$

Surface S enclosing volume V





flux going out of S_1 = flux going into S_2

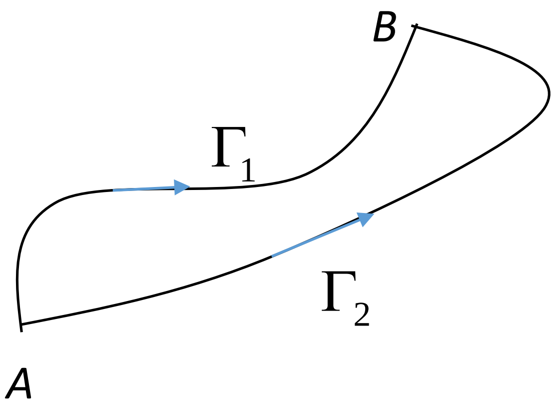
Recall that divergence of a vector field multiplied by the volume is equal to the flux out of the volume:

$$d\vec{S}\cdot\vec{F}(\vec{r}) = dV\ \nabla\cdot\vec{F}(\vec{r})$$

When sum over the small volume segments, contribution from a surface of two neighbouring volumes cancels out.
→ Only outer surface matters (in analogy to Stokes formula, in which only the outer loop matters)

Conservative field

A vector field is called conservative if
the work done by the field from point *A* to point *B* is ***independent of the path***.



$$\int_{\Gamma_1} d\vec{r}\cdot\vec{F}(\vec{r}) = \int_{\Gamma_2} d\vec{r}\cdot\vec{F}(\vec{r})$$

If the field is conservative then the work done around a closed loop is zero
because the work done from *A* to *B* is the negative of the work done from *B* to *A*.
In other words, going from *A* to *B* and then back to *A* is the same as going from A to *A*,
i.e., not moving at all so that there is no work done.

A conservative field implies that it can be obtained as a gradient of a scalar field:

$$\vec{F}(\vec{r}) = -\nabla \phi(\vec{r})$$

Check that the work done is independent of the path:

$$\int_A^B d\vec{r} \cdot \vec{F}(\vec{r}) = -\int_A^B d\vec{r} \cdot \nabla \phi(\vec{r})$$

$$= -\int_A^B \left(dx \frac{\partial \phi}{\partial x} + dy \frac{\partial \phi}{\partial y} + dz \frac{\partial \phi}{\partial z} \right)$$

$$= -\int_A^B d\phi$$

$$= -\left[\phi(B) - \phi(A) \right]$$

A conservative field also implies that $\nabla \times \vec{F}(\vec{r}) = 0$

This follows from the mathematical identity $\nabla \times (\nabla \phi) = 0$

It can also be understood from Stokes theorem:

$$\oint_{C(S)} d\vec{r} \cdot \vec{F}(\vec{r}) = \int_{S(C)} d\vec{S} \cdot (\nabla \times \vec{F}) = 0$$

because the work done around a closed loop is zero for a conservative field.

True for any surface *S(C)* so that $\nabla \times \vec{F}(\vec{r}) = 0$

Defining a line or curve in 3D

A line in three-dimensional space can be defined by a parameter λ, with value from *0* to *1*.

$$\vec{r}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$$

As λ is varied from *0* to *1*, the vector position ***r***(λ) traces a curve in space.

Alternatively, one can eliminate λ and uses one of the coordinates as a parameter.

Examples:

$\vec{r}(\lambda) = (\lambda, \lambda^2, 0)$ $\quad$ $\vec{r}(x) = (x, x^2, 0)$ This defines a parabola $y = x^2$ *in* x-y plane.

$\vec{r}(\lambda) = (\lambda, 2\lambda, \lambda^2)$ $\quad$ $\vec{r}(x) = (x, 2x, x^2)$ This defines a curve in 3D with $y=2x$ and $z=x^2$

Differential:

$$d\vec{r}(\lambda) = (dx(\lambda), dy(\lambda), dz(\lambda)) = \left(\frac{dx}{d\lambda}, \frac{dy}{d\lambda}, \frac{dz}{d\lambda}\right)d\lambda$$

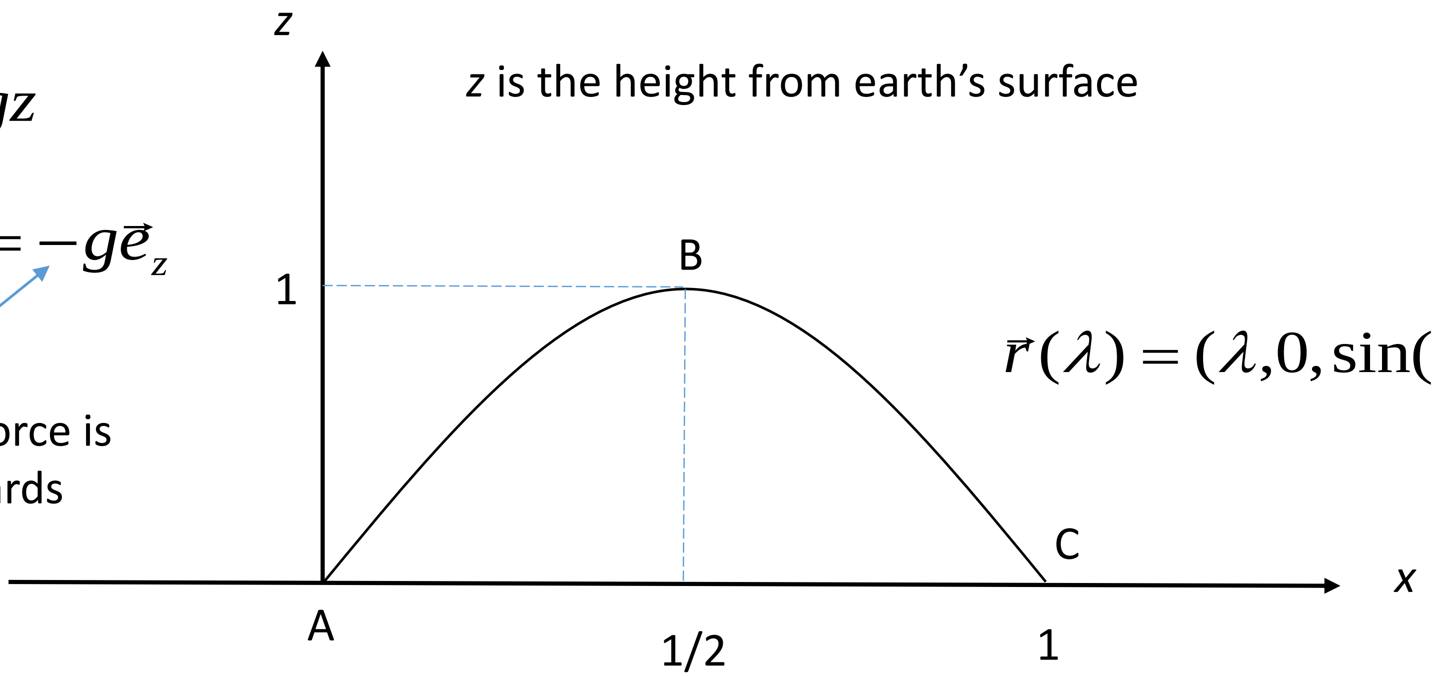
Example of conservative field: gravitational field

Gravitational potential: $\phi(z) = gz$

Gravitational force $\vec{F} = -\nabla\phi(z) = -g\vec{e}_z$

Negative sign means the force is directed downwards, towards the earth's surface

z is the height from earth's surface



$$\vec{r}(\lambda) = (\lambda, 0, \sin(\pi\lambda))$$

Work done *by the field*

$$W = \int_A^B d\vec{r}\cdot\vec{F}(\vec{r}) = -g\int_A^B d\lambda\left(\frac{dx}{d\lambda}\vec{e}_x + \frac{dy}{d\lambda}\vec{e}_y + \frac{dz}{d\lambda}\vec{e}_z\right)\cdot\vec{e}_z$$

$$= -g\int_A^B d\lambda\,\frac{dz}{d\lambda}$$

$$= -g\int_0^1 dz = -g\,z\Big|_0^1 = -g$$

The work done by a person climbing up the hill from A to B is then $-W = g$ (the negative of the work done by the field). Since the field is conservative, the work done is also given by

$$W = -[\phi(B) - \phi(A)] = -g$$